# 5. Root Locus Design

## Electronic Control Systems

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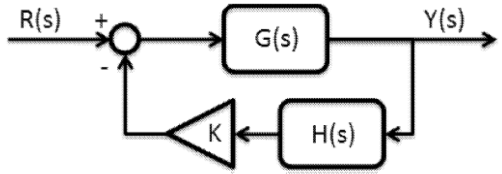
# Locus of the Roots (closed loop poles)

- General Feedback Control System




- Transfer Function

$$\begin{aligned} Y(s) &= G(s)\{R(s) - KH(s)Y(s)\} \\ Y(s)\{1 + KH(s)G(s)\} &= G(s)R(s) \\ \frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + KH(s)G(s)} \end{aligned}$$

- Root Locus

$$\begin{aligned} 1 + KH(s)G(s) &= 0 \\ KH(s)G(s) &= -1 \end{aligned}$$

# Locus of the Roots (closed loop poles)

- Magnitude and Phase conditions

$$\begin{aligned} |KG(s)H(s)| &= 1 \\ \angle KH(s)G(s) &= \pm 180^o(2k+1) \quad \text{where } k = 0, 1, 2, \ldots \end{aligned}$$

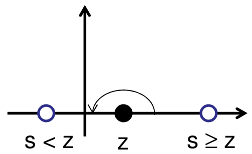
- Phase condition

$$\begin{aligned} \angle K\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} &= \pm 180^o(2k+1) \\ \angle(s-z_1) + \angle(s-z_2) + \ldots + \angle(s-z_m) & \\ -\angle(s-p_1) - \angle(s-p_2) - \ldots - \angle(s-p_n) &= \pm 180^o(2k+1) \end{aligned}$$

# Phase of Zeros and Poles

**Angle of a real zero on the real axis**

$$\angle(s-z) = \begin{cases} 0 & \text{if} \quad s \geq z \\ 180^o & \text{if} \quad s < z \end{cases}$$




**Angle of a real pole on the real axis**

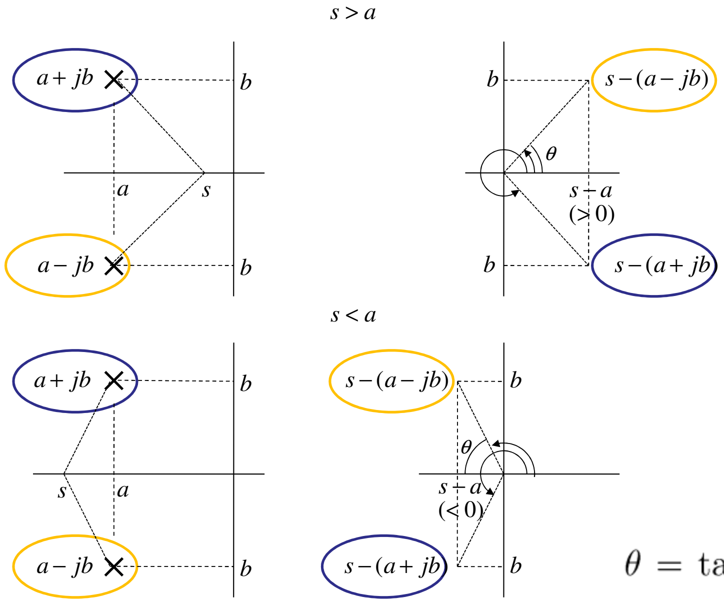
$$-\angle(s-p) = \begin{cases} 0 & \text{if} \quad s \geq p \\ -180^o & \text{if} \quad s < p \end{cases}$$

**Angle on the real axis by a complex conjugate pair of poles**

$$\begin{aligned} & -\angle(s-a+jb) \\ & -\angle(s-a-jb) = \begin{cases} -(360^o - \theta) - \theta = -360^o \\ \text{if} s \geq a \\ -(180^o + \theta) - (180^o - \theta) = -360^o \\ \text{if} s < a \end{cases} \end{aligned}$$

No phase contribution on the real axis

# Conjugate pair of Poles

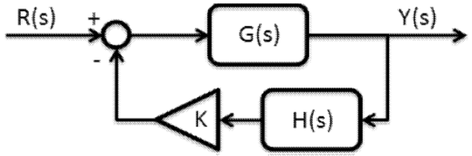



$\theta = \tan^{-1} \frac{b}{|s-a|}$

# Root Locus Rules

**Rule 1: Root locus is on the real axis to the left of an odd number of poles or zeros**

**Rule 2: Root locus starts from open loop poles ($K \to 0$), and approach open loop zeros or infinity ($K \to \infty$)**




$$K \to 0 \; G_c(s) \to G(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2)...(s-z_m) \Leftarrow \text{OL zeros}}{(s-p_1)(s-p_2).....(s-p_3) \Leftarrow \text{OL poles}}$$

$$K \to \infty \quad R(s) - KY(s) \to -KY(s)$$

R(s) becomes insignific ant, and the system becomes autonomous ,

Then $-KY(s)G(s) = Y(s)$

$$-K = \frac{1}{G(s)} = \frac{D(s)}{N(s)}; \quad \text{as } K \to \infty \; N(s) \to 0 \text{ or } D(s) \to \infty$$

# Root Locus Rules

**Rule 3: Root locus asymptotes, asymptote angles, and point of intersection**

Root locus needs to have $n - m$ number of asymptotes. Those poles that reach infinity as $K$ increases will reach infinity along these asymptotes. As $m$ number of poles reach the open loop zeros as $K \to \infty$ the remaining number of poles is $n - m$, and therefore, same number of asymptotes are required.

**Asymptote Intersection point**

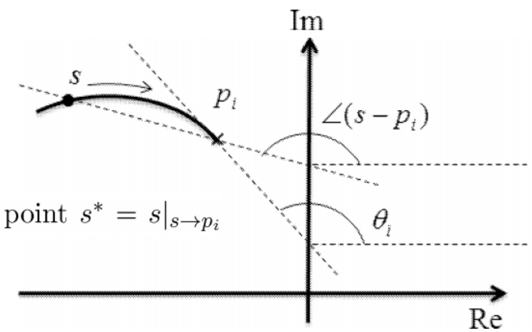
$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

**Asymptote angles**

$$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n-m} \; ; l = 0,1...(n-m-1)$$

# Root Locus Rules

**Rule 4: Angle of departure/arrival**




A very close point $s^* = s|_{s \to p_i}$

$$\angle G(s)|_{s^*} = \pm 180^o(2k+1)$$

$$\angle(s^*-z_1) + \angle(s^*-z_2) + \ldots + \angle(s^*-z_m)$$
$$-\angle(s^*-p_1) - \angle(s^*-p_2) - \ldots - \angle(s^*-p_i)$$
$$- \ldots - \angle(s^*-p_n) = \pm 180^o(2k+1)$$

# Root Locus Rules

$$\angle(p_i - z_1) + \angle(p_i - z_2) + \ldots + \angle(p_i - z_m) - \angle(p_i - p_1) - \angle(p_i - p_2) - \ldots - \theta_i - \ldots - \angle(p_i - p_n) \approx \pm 180^o(2k+1)$$

**Rule 5: Break away points**

$$\frac{d}{ds}\left\{\frac{1}{H(s)G(s)}\right\} = 0$$

$$\frac{d}{ds}\left\{\frac{D_{HG}(s)}{N_{HG}(s)}\right\} = 0 \qquad H(s)G(s) = \frac{N_{HG}(s)}{D_{HG}(s)}$$

**Rule 6: Stability margin**

The coefficients of the characteristic equation $\Delta(s)$ are arranged in the Routh array, which determines the range of gain $K$ for stable response. An example will be worked out to demonstrate the Routh stability criterion.

# Root Locus Rules

**Rule 7: Gain calculation**

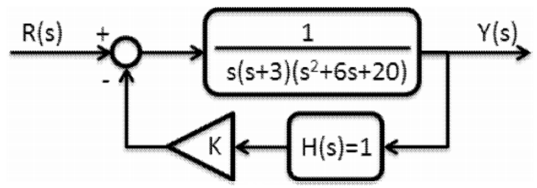
$$K^* = \frac{1}{|H(s)G(s)|_{s=s^*}}$$

$$K^* = \left|\frac{D_{HG}(s)}{N_{HG}(s)}\right|_{s=s^*}$$

Gain K for a given point on the root locus

### 4.2.2 Example

Draw the root locus of the following feedback control system




# Example: Root Locus Design

Determine

1. Asymptotes and asymptote angles
2. Angle of departure
3. Break away point
4. Maximum stable gain
5. Forward gain for unity DC gain when $K = 30$

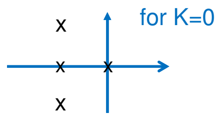
Answer

- Open loop Poles

The characteristic equation is $1 + K\frac{1}{s(s+3)(s^2+6s+20)} = 0$

$K = 0, \quad s(s+3)(s^2+6s+20) = 0$

$0, -3, -3 \pm j\sqrt{11}$




**n=4 (4 poles) and m=0 (no zeros)**

# Example: Root Locus Design

- Asymptotes

$$\phi_0 = \frac{180^0 + 360^0(0-1)}{4} = -45^0$$

$$\phi_1 = \frac{180^0 + 360^0(1-1)}{4} = 45^0$$

$$\phi_2 = \frac{180^0 + 360^0(2-1)}{4} = 135^0$$

$$\phi_3 = \frac{180^0 + 360^0(3-1)}{4} = 225^0$$

- Asymptote intersection point

$$\alpha = \frac{(0) + (-3) + (-3 + j\sqrt{11}) + (-3 - j\sqrt{11})}{4} = -2.25$$

# Example: Root Locus Design

- Angle of departure from pole $-3+j\sqrt{11}$

$$
\begin{aligned}
-\angle(-3+j\sqrt{11}-0)-\angle(-3+j\sqrt{11}+3) & \\
-\theta-\angle(-3+j\sqrt{11}-(-3-j\sqrt{11})) &= 180^0(2k+1)\\
-\left(180^0-\tan^{-1}\frac{\sqrt{11}}{3}\right)-90^0-\theta-90^0 &= 180^0(2k+1)\\
48^0-\theta &= 180^0\\
\theta &= -132^0
\end{aligned}
$$

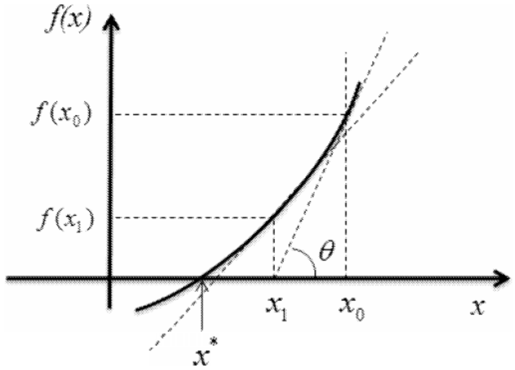
- Breakaway Point

$$
\begin{aligned}
\frac{d}{ds}\left\{\frac{1}{H(s)G(s)}\right\} &= 0\\
\frac{d}{ds}\{s(s+3)(s^2+6s+20)\} &= 0\\
\frac{d}{ds}\{s^4+9s^3+38s^2+60s\} &= 0\\
4s^3+27s^2+76s+60 &= 0
\end{aligned}
$$

$f(-2)<0$ and $f(-1)>0$

$s_0$ =-1.5

# Newton-Raphson Method




$$
\begin{aligned}
f(x_0) &= (x_0-x_1)f'(x_0)\\
x_1 &= x_0-\frac{f(x_0)}{f'(x_0)}
\end{aligned}
$$

# Example: Root Locus Design

$$
\begin{aligned}
s_1 &= s_0-\frac{f(s_0)}{f'(s_0)}\\
&= s_0-\frac{4s^3+27s^2+76s+60}{12s^2+54s+76}\Big|_{s_0}\\
&= -1.5-\frac{4\times-1.5^3+27\times-1.5^2+76\times-1.5+60}{12\times-1.5^2+54\times-1.5+76}\\
&= -1.5-\frac{-6.75}{22}\\
&= -1.19\\
s_2 &= s_1-\frac{f(s_1)}{f'(s_1)}\\
&= -1.19-\frac{4\times-1.19^3+27\times-1.19^2+76\times-1.19+60}{12\times-1.19^2+54\times-1.19+76}\\
&= -1.19-\frac{1.05}{28.73}\\
&= -1.19-0.04\\
&= -1.23
\end{aligned}
$$

# Example: Root Locus Design

- Gain at the breakaway point $K = |D_{HG}(s)|_{s=-1.23}$

$$|-1.23(-1.23+3)(-1.23^2+6\times-1.23+20)| = |-30.77| \approx 30.8$$

- Stability margin $\Delta(s) = s(s+3)(s^2+6s+20)+K = 0$

Routh Array $\Delta(s)=s^4+9s^3+38s^2+60s+K=0$

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $38$ | $K$ |
| $s^3$ | $9$ | $60$ | $0$ |
| $s^2$ | $\frac{9\times38-60\times1}{9}=31.3$ | $\frac{60K-0\times38}{60}=K$ | $0$ |
| $s^1$ | $\frac{31.3\times60-9\times K}{31.3}=60-0.29K$ | $0$ | $0$ |
| $s^0$ | $\frac{(60-0.29K)\times K-0\times31.3}{60-0.29K}=K$ | $0$ | $0$ |

for stability $60-0.29K\geq0$, i.e. $K\leq208$

# MatLab: Root Locus Design

```
% Root Locus of 1+KH(s)G(s)=0 H(s)=1

% Plant OLTF
num=[1]; den=conv([1 0],conv([1 3],[1 6 20]));
G=tf(num,den);

% Draw root locus
rlocus(G);
grid on;
```

# MatLab: Root Locus Design